G. Peano and M. Gramegna on ordinary differential equations

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AT THE ORIGINS OF FUNCTIONAL ANALYSIS: G. PEANO AND M. GRAMEGNA ON ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. — In the context of the theory of infinite matrices and linear operators, two articles by Peano and by Gramegna on systems of linear differential equations have interesting implications for the reconstruction of research on functional analysis between 1887 and 1910. With the aim of evaluating their historical value, linked to logic and vector calculus, this paper provides a detailed analysis of their treatment, demonstrating the modernity of the analytic tools used. In this paper we also reveal the negative consequences that Gramegna's note had on Peano's lectureship in Higher Analysis, leading to his dismissal, which marked the beginning of the progressive decline of his school.

Résumé (À l'origine de l'analyse fonctionnelle : G. Peano et M. Gramegna à propos des équations différentielles ordinaires)

Dans le cadre de la théorie des matrices infinies et des opérateurs linéaires, deux articles de Peano et de Gramegna respectivement sur les systèmes d'équations différentielles linéaires présentent des aspects intéressants, qui permettent de reconstituer les recherches d'analyse fonctionnelle entre 1887 et 1910. Visant à évaluer ces deux écrits d'un point de vue historique, cet article en donne une description détaillée et met en valeur la modernité des instruments analytiques utilisés. Seront également mises en avant les conséquences négatives que la recherche de Gramegna a eues sur le cours d'analyse supérieure de Peano à l'université de Turin, à savoir le congédiement de Peano et le déclin progressif de son école.

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1. GIUSEPPE PEANO AND MARIA GRAMEGNA

The history of the theory of infinite matrices and linear operators defined on function spaces is of great importance in the development of functional analysis[1]. It began at the end of the 19th century with the studies of H. Poincaré on infinite systems of linear equations and was continued by H. von Koch, who alluded to the theory of infinite determinants in 1893[2]. A great impetus for research in this field came between 1904 and 1910 from D. Hilbert, who attempted to study integral equations previously investigated by I. Fredholm in 1900[3]. This field was further developed in the years 1906-1927 by mathematicians – including E. Schmidt, F. Riesz, E. Hellinger and O. Toeplitz – who found many fundamental theorems of the theory of abstract operators, although they were expressed in matrix terms[4]. In particular the function space concept (merely implied in Hilbert's works) was given a concrete form by Schmidt, who seemed to be closer to the modern theory of function spaces and linear operators and showed an acquaintance with the notions of vectors and space[5]. Another strand of these developments was based in Italy, where G. Ascoli, C. Arzelà and V. Volterra worked with sets of functions and where, from another point of view, G. Peano and S. Pincherle were leaders in studies on functionals (*funzionali*) and *distributive operations*, referring to the tradition of symbolic calculus associated with the names of H. Grassmann and E. Laguerre. Their works, dated between 1888 and 1901, in turn inspired

[1] Cf. [Bourbaki 1960], [Bernkopf 1966-67, pp. 1-96], [Bernkopf 1968, pp. 308-358], [Monna 1973], [Dieudonné 1978, vol. 2, pp. 115-176], [Dieudonné 1981], [Siegmund-Schultze 1982, pp. 13-71] and [Kline 1991, pp. 1227-1277]. There are also suggestive historical backgrounds in [Lévy 1922, pp. 1-10], [Fréchet 1928, p. 21] and [De Giorgi 1989, pp. 41-60].

[2] Cf. [Bourbaki 1960, pp. 225-226], [Bernkopf 1966-67, pp. 3-5], [Bernkopf 1968, pp. 316-327], [Monna 1973, pp. 11-14], [Dieudonné 1978, vol. 2, pp. 150-153] and [Dieudonné 1981, pp. 77-79].

[3] Cf. [Bourbaki 1960, pp. 226-228], [Bernkopf 1966-67, pp. 9-33], [Bernkopf 1968, pp. 327-331], [Monna 1973, pp. 15-17, pp. 29-37, pp. 58-62], [Dieudonné 1978, vol. 2, pp. 155-159], [Dieudonné 1981, pp. 105-115] and [Kline 1991, pp. 1234-1248].

[4] Cf. [Bourbaki 1960, p. 228, pp. 229-234], [Bernkopf 1966-67, pp. 44-62], [Bernkopf 1968, pp. 331-337], [Monna 1973, pp. 17-21, pp. 37-41, pp. 62-70], [Dieudonné 1978, vol. 2, pp. 159-160, p. 163, pp. 165-169], [Dieudonné 1981, pp. 117-120, pp. 122-135, pp. 144-160] and [Kline 1991, pp. 1248-1252].

[5] Cf. [Monna 1973, p. 19, p. 39] and [Kline 1991, pp. 1262-1264].

J. Hadamard and M. Fréchet. Fréchet provided a new insight on function spaces by giving an additional geometric structure to the space concept, with the introduction of the fundamental notion of abstract metric space[6]. However, for many years, until the independent rediscovery – by E.H. Moore in 1910 and by the French around 1928 – of the Italian contributions on abstract linear spaces, analyzed with an intrinsic approach, these appeared to have been ignored and many mathematicians continued thinking in terms of matrices[7]. The work of Peano and Gramegna on systems of linear differential equations provide us with a means to analyse the link between research in abstract algebra and in functional analysis during the period of transition from the matrix approach to the operator-theoretic viewpoint.

Maria Gramegna was a student of Giuseppe Peano[8], while he was Professor of Infinitesimal Calculus and of Higher Analysis at the University of Turin, where he taught for over fifty years. Between 1891 and 1908 Peano devoted his energies to the edition of the *Formulaire de mathématiques*, an encyclopaedic work in symbolic language, which includes results on classical mathematics. In Turin, he moulded a large number of scholars, many of whom collaborated with him, among them Maria Paola Gramegna. As she is not well known in history of mathematics, we provide a short biography.

Maria Gramegna, daughter of Maria Cristina Agosta and Innocenzo Gramegna, was born in Tortona (Alessandria) on 11 May 1887. She attended secondary school at the Royal Lyceum Severino Grattoni in Voghera, where the mathematician G. Vitali was teaching, and she enrolled at Turin University on 14 November 1906. She earned the *licenza* in 1908. For four academic years (1906–1910) she won a room-and-board

[6] Cf. [Bourbaki 1960, pp. 84-85, pp. 228-229], [Bernkopf 1966-67, pp. 34-44], [Monna 1973, pp. 41-48, p. 51, pp. 55-56, pp. 114-127], [Dieudonné 1978, vol. 2, pp. 160-163, pp. 169-171], [Dieudonné 1981, pp. 116-117, pp. 121-122, pp. 144-160] and [Kline 1991, pp. 1256-1260].

[7] Cf. [Bernkopf 1966-67, pp. 44-46], [Monna 1973, pp. 127-132], [Dieudonné 1978, vol. 2, pp. 210-216], [Dieudonné 1981, p. 212, p. 216] and [Siegmund-Schultze 1998, pp. 51-89].

[8] Giuseppe Peano was born in Spinetta, near Cuneo, on 27 August 1858 and died in Turin on 20 April 1932. He graduated from Turin University in 1880, and in 1890 became Professor of Infinitesimal Calculus. Between 1925 and 1932 he changed his position to the chair in Mathematics Education. From 1908 to 1910 Peano was also appointed to teach the course in Higher Analysis.

scholarship at the *Collegio delle provincie 'Carlo Alberto'*, which had been established to assist students who came from the provinces to attend the University[9]. Peano gave her the 'highest honour' as his evaluation of the examination in Higher Analysis[10].

In the meeting of the Turin Academy of Sciences of 13 March 1910, four months before Gramegna graduated, Peano submitted her note entitled "*Serie di equazioni differenziali lineari ed equazioni integro-differenziali*" [Gramegna 1910, pp. 469-491]. It was published in the *Proceedings* of the Academy and submitted by Gramegna, under the same title, as her graduation thesis in Mathematics[11]. On 7 July 1910 she graduated with high honours (110/110) and, on 19 July, she took the final examination at the *Scuola di Magistero*, a teacher-training institution, with the dissertation "*Area della zona sferica e della sfera*".

In 1911, upon winning a competition for teaching positions, Gramegna went to teach in Avezzano (L'Aquila), where she held an appointment at the Royal Normal School[12]. She became headmistress of the Municipal College. Two years later, she was offered a position in Piacenza – as she had originally requested – but she turned the offer down, preferring to stay in Avezzano by then.

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[9] Cf. [ASUT-XI-7; ASUT-XI-22; ASUT-XI-31].

[10] Peano had several female students in his courses of Infinitesimal Calculus and Higher Analysis. Among them, we can recall Margherita Peyroleri, Maria Gramegna, Rosetta Frisone, Virginia Vesin, Paolina Quarra, Maria Destefanis, Elisa Viglezio, Piera Chinaglia, Maria Cibrario and Fausta Audisio. Only Maria Gramegna and Maria Cibrario obtained the highest evaluation for both the exams of Infinitesimal Calculus and Higher Analysis. Cf. [ASUT-58; ASUT-63].

[11] For her final examination Gramegna discussed three short dissertations as well, dealing with Higher Geometry, Geodesy and Hydrostatic: "*Movimenti a traiettorie sferiche*", "*Osservazioni sulle equazioni dell'idrostatica e sulle congruenze di curve*" and "*Differenza fra la lunghezza del perimetro di una sezione normale perpendicolare al meridiano in più punti di latitudine φ e la intera geodetica che inviluppa il parallelo di latitudine φ*".

[12] Cf. [ASA]. Gramegna always remained in contact with her teacher Peano. In his article "*Le definizioni in matematica*", Peano thanked Gramegna for having helped him in research on history of mathematics. Cf. [Peano 1911, p. 70]: "The historical background and the quotations from Aristotle and Pascal were furnished to me by Dr. Maria Gramegna, now a teacher in Avezzano, and formerly a student at Turin University" ("*Il materiale storico e le citazioni di Aristotele e di Pascal mi furono fornite dalla Dott. Maria Gramegna, ora Prof. in Avezzano, e prima studente all'Università di Torino*").

On 13 January 1915 she died victim of the earthquake that destroyed Avezzano. On 24 January Peano wrote to his friend and colleague, R. Marcolongo, at Naples University, expressing deep sadness :

> "Unfortunately the news is true. Gramegna didn't leave Avezzano, where she was headmistress of the School and of the Municipal College. She persuaded her old mother to go to live with her. Both died, victims of the earthquake. Her two brothers, who lived in their home town Tortona, went to Avezzano and came back without learning anything about the sister or the mother. She was loved by the entire town and this is the reason why she didn't leave Avezzano. The poor girl!"[13].

2. PEANO ON SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

The analysis of two papers on systems of linear differential equations, written by Peano and by Gramegna respectively, permits us to understand the actual nature of the analytic research carried out by Peano's school through the first decade of the twentieth century.

In the winter of 1887 Peano started to study differential systems and submitted a note entitled "*Integrazione per serie delle equazioni differenziali lineari*" at the Turin Academy of Sciences [Peano 1887, pp. 437-446]. A slightly modified version of this paper was published the following year, in French, by *Mathematische Annalen* [Peano 1888b, pp. 450-456].

This work is one of a series of notes published between 1885 and 1897, in which Peano deals with differential equations in a rigorous way[14]:

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[13] G. Peano to R. Marcolongo, Turin 24.1.1915 *in* [Roero 2006, p. 39]: "*Purtroppo la cosa è come Ella scrive. La Maria Gramegna rimase ad Avezzano, ove fu nominata direttrice della scuola, o del convitto. Fece venire a coabitare seco la vecchia madre. E amendue rimasero sotto le macerie del terremoto. Due fratelli, residenti in Tortona, loro paese nativo, si recarono ad Avezzano, e ritornarono senza notizie della sorella e della madre. Ella era amata da tutto il paese, e questa è la ragione per cui non si era mossa di là. Povera signorina!*".
A. Terracini, who was a fellow student of Gramegna, and later professor at Turin University, wrote in his autobiography *Ricordi di un matematico* : "During the second year, I was a fellow student of Maria Gramegna. She wrote a paper entitled *Serie di equazioni lineari ed equazioni integro-differenziali* (Atti Acc. Sc. Torino, vol. XLV, 1910). She died some years later, in 1915, a victim of the earthquake that struck Avezzano" ("*Fu anche mia compagna in second'anno Maria Gramegna, la quale lasciò un lavoro* Serie di equazioni lineari ed equazioni integro-differenziali *(Atti Acc. Sc. Torino, vol. XLV, 1910). Essa morì qualche anno dopo, nel 1915, tra le vittime del terremoto di Avezzano*" [Terracini 1968, p. 12]). About Gramegna's death, cf. also *Il Bollettino di Matematica* (Conti), 13, 1914-1915, p. XII.

[14] [Peano 1885, pp. 677-685], [Peano 1887, pp. 437-446], [Peano 1888b, pp. 450-456], [Peano 1890, pp. 182-228], [Peano 1892, pp. 79-82], [Peano 1894, p. 136] and [Peano 1897, pp. 9-18].

in 1886 he provided the proof of the so-called Cauchy-Peano theorem on the existence of the solutions for a differential equation and he extended this theorem to systems of differential equations in 1890. The first drastic refusal of the axiom of choice by Peano is noteworthy in this context[15]. In these works Peano applied, for the first time, the methods and language of his mathematical logic; this use of symbolical procedures delayed the reception of his articles by the international mathematical community. Only after the translation into ordinary mathematical language by G. Mie in 1893 of Peano's article "*Démonstration de l'intégrabilité des équations différentielles ordinaires*" [Peano 1890, pp. 182-228], did the importance of his results and the elegance of his proofs become recognized[16].

Despite this, Peano considered his approach very important, as he wrote to C. Jordan in 1894:

> "For the first time mathematical logic was applied to the analysis of a problem in the field of the higher mathematics; and this application is, in my opinion, the most important aspect of my work. Nevertheless it will take time for the symbols and the procedures of the logic to be comprehended, and my proof is not well-known. Mie published an explanatory article in the *Mathematische Annalen*, Bd. 43, p. 553. But later on many papers were published on the same topic that did not add anything new (except for some inaccuracies) and did not mention my work. This displeases me, because I believe that mathematical logic will be advantageous in the analysis of difficult problems"[17].

Together with his students and other scholars, Peano began the research program of the *Formulario* (1891-1908), which includes mathematical logic, analysis, arithmetic and geometry, which he was to pursue until the end of his life [Cassina 1955, pp. 244-265, pp. 544-574].

In the note "*Integrazione per serie delle equazioni differenziali lineari*" Peano applied to the study of differential systems the method of 'successive approximations' or, as he preferred to call it, 'successive integrations'.

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[15] Cf. [Moore 1982, pp. 76-82] and [Cassinet & Guillemot 1983, pp. 77-80, pp. 261-266].

[16] Cf. [Mie 1893, pp. 553-568] and [Ghizzetti 1986, p. 57].

[17] G. Peano to C. Jordan, Turin 6.11.1894 *in* [Borgato 1991, p. 96]: "*È la prima volta che si è applicata la logica matematica all'analisi di una questione di matematiche superiori; e quest'applicazione è, secondo me, la cosa più importante del mio lavoro. Ma i simboli e le operazioni della logica necessitano di tempo per essere appresi e la mia dimostrazione è poco conosciuta. Mie ha pubblicato un articolo esplicativo sui* Mathematische Annalen*, Bd. 43, pag. 553. Ma in seguito sono apparsi molti lavori sullo stesso soggetto, senza aggiungere nulla di nuovo (salvo qualche inesattezza), e senza far menzione del mio lavoro. Ciò mi dispiace, perchè credo che la logica matematica apporterà grandi vantaggi nell'analisi delle questioni difficili*".

The purpose of Peano's article is to prove the following theorem: let n homogeneous linear differential equations in n functions $x_1, x_2, \ldots, x_n$ of a real variable t, in which the coefficients α_{ij} are functions of t, continuous on a closed and bounded interval $[p, q]$ be expressed as:

$$
(1) \qquad \begin{cases} \dfrac{dx_1}{dt} = \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n \\ \dfrac{dx_2}{dt} = \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n \\ \qquad\qquad \vdots \\ \dfrac{dx_n}{dt} = \alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nn}x_n \end{cases} .
$$

Substituting n arbitrary constants $a_1, a_2, \ldots, a_n$ on the right side of the equations, in place of $x_1, x_2, \ldots, x_n$, and integrating from t_0 to t, with $t_0,\ t \in (p, q)$, we obtain n functions of t, denoted by $a'_1, a'_2, \ldots, a'_n$. Now substitute $a'_1, a'_2, \ldots, a'_n$ in the right side of the given equations in place of $x_1, x_2, \ldots, x_n$ and integrate from t_0 to t. We get n new functions of t, denoted by $a''_1, a''_2, \ldots, a''_n$. Repeating this procedure, we obtain:

$$
(2) \qquad \begin{cases} a_1 + a'_1 + a''_1 + \ldots \\ a_2 + a'_2 + a''_2 + \ldots \\ \qquad \vdots \\ a_n + a'_n + a''_n + \ldots \end{cases} .
$$

These series are convergent over the interval (p, q). Their sums, which we shall indicate by $x_1, x_2, \ldots, x_n$, are functions of t which satisfy the given system. Moreover, for $t = t_0$, they assume the arbitrarily chosen values $a_1, a_2, \ldots, a_n$.

To prove this theorem, Peano introduced vector and matrix notation and he used the theory of *complexes* and *transformations*, developed by H. Grassmann, W. Hamilton, A. Cayley and J. Sylvester, although in a different way. These new techniques of geometric calculus, then at an early stage, along with the study of the method of successive approximations sketched by J. Caqué and L. Fuchs, led Peano to a new proof of the existence of a solution to a system of linear differential equations[18]. He

[18] Cf. [Grassmann 1844], [Cayley 1887], [Hamilton 1853], [Sylvester 1852], [Caqué 1864] and [Fuchs 1870].

introduced therein new concepts in a rigorous and clear way, namely the definition of matrix-valued exponential functions.

Let us look at the crucial passages of this work. First, Peano develops the basis of linear algebra in the space of *complex numbers of order* n, which are n-tuples of real numbers $\overline{a} = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^n$[19]. He defines for these *complexes* the relation of *equality*, the *sum* and the *scalar product*, proving the associative, commutative and distributive properties by a 'component by component' argument. He then defines the *linear combination* $k_1\overline{a}_1 + k_2\overline{a}_2 + \cdots + k_n\overline{a}_n$ where $k_i \in \mathbb{R}$, $\overline{a}_i \in \mathbb{R}^n$, for $i = 1, \ldots, n$, and for finite dimensional spaces he states the existence of the canonical basis.

Peano defines the *modulus of a complex*, which corresponds to the Euclidean norm of $\overline{a} \in \mathbb{R}^n$:

$$\|\overline{a}\| = \operatorname{mod} \overline{a} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}, \tag{3}$$

and he shows the properties that are now used to define a norm on $\mathbb{R}^n$.

Finally he states some basic notions of analysis in the space of complexes, naturally extending the usual definitions of *limit*, *derivative* and *integral*. For example, on the concept of limit, Peano remarks:

> "We shall say that the variable complex $\overline{x} = [x_1, x_2, \ldots, x_n]$ has as limit the complex $\overline{a} = [a_1, a_2, \ldots, a_n]$, if $\lim x_1 = a_1$, $\lim x_2 = a_2$,.. $\lim x_n = a_n$. We deduce that, if $\lim \overline{x} = \overline{a}$, then $\lim \|\overline{x} - \overline{a}\| = 0$ and vice versa. Having defined the sum and the limit of complexes, we may extend the definition of convergence to series of complexes. We can prove that a series of complexes is convergent if the series formed by their moduli is convergent" [20].

[19] The term 'complex' is introduced as follows: "We will call *complex number* of order n the set of n real numbers. The complex that is formed by the numbers $a_1, a_2, \ldots, a_n$ will be indicated by $[a_1, a_2, \ldots, a_n]$; if there is no need of giving prominence to the real numbers that compose the complex, this will be indicated by a single letter $\overline{a}$" ("*Diremo* numero complesso *di specie n l'insieme di n numeri reali. Il complesso formato dai numeri* $a_1, a_2, \ldots, a_n$ *si indicherà con* $[a_1, a_2, \ldots, a_n]$*; quando non occorra di mettere in evidenza i numeri reali che compongono il complesso, lo si indicherà con una sola lettera* $\overline{a}$"). Peano does not yet use the word 'space' but he shows to be acquainted with this notion. About the philological significance of the term 'complex' and its use, cf. [Peano 1908, p. 144, p. 161]. On this subject there is a misunderstanding in [Hahn & Perazzoli 2000, p. 503].

[20] [Peano 1887, p. 440]: "*Diremo che il complesso variabile* $\overline{x} = [x_1, x_2, \ldots, x_n]$ *ha per limite il complesso* $\overline{a} = [a_1, a_2, \ldots, a_n]$ *se* $\lim x_1 = a_1$, $\lim x_2 = a_2, \ldots \lim x_n = a_n$. *Si deduce che, se* $\lim \overline{x} = \overline{a}$, $\lim \|\overline{x} - \overline{a}\| = 0$, *e viceversa. Essendo definiti per i complessi la somma ed il limite, si può estendere alle serie a termini complessi la definizione di convergenza. Si dimostra che una serie a termini complessi è convergente se è convergente la serie formata coi loro* moduli".

The second part of the article deals with the theory of *transformations*, or linear operators, defined on $\mathbb{R}^n$ and represented by the matrix $\alpha = (\alpha_{ij})_{i,j=1,\dots,n} \in \mathscr{M}_n(\mathbb{R})$. Peano develops the elementary properties of these operators, the space of which is here denoted by $\mathscr{L}(\mathbb{R}^n)$, introducing the *equality*, the *sum* and the *product*, (by the now usual matrix multiplication method). This basic exposition is expounded in the ninth chapter "*Trasformazioni di sistemi lineari*" of his *Calcolo Geometrico secondo l'*Ausdehnungslehre *di Hermann Grassmann, preceduto dalle operazioni della logica deduttiva* [Peano 1888a, pp. 141-152] [21]. Here we find a "modern" axiomatic definition of the concept of linear spaces, and the idea of linear transformation is introduced in a direct way, without the use of matrices.

The last section of the note [Peano 1887] includes some preliminary studies of functional analysis on linear operators. Peano defines the *modulus of the transformation* α, that is the operator norm on $\mathscr{L} = \mathscr{L}(\mathbb{R}^n)$:

$$\|\alpha\|_{\mathscr{L}} = \sup_{\bar{x}\neq 0} \frac{\|\alpha\bar{x}\|}{\|\bar{x}\|}, \tag{4}$$

and defines the *convergence* with respect to this norm. He introduces the *exponential of a transformation* e^A and discusses the rapid convergence of its Taylor series. Independently, E. Carvallo [1891, p. 187, pp. 227-228] obtained the same result some years later. Peano claimed his priority in "*Sur les systèmes linéaires*" with the words:

> " M. Carvallo a publié dans ce journal [...] des notes de la plus grande importance. Mais il y a deux lacunes, qu'il convient de combler. À la pag. 11, il admet pour e^φ, lorsque φ est un système linéaire, la définition $e^\varphi = 1+\varphi+\frac{\varphi^2}{2!}+\dots$ et il ajoute "Il resterait à montrer que ce développement est convergent". Or j'ai donné cette démonstration dans ma note *Intégration par série des équations différentielles linéaires* (Atti Acc. Torino, 1887, et Mathematische Annalen)" [Peano 1894, p. 136].

The novelty of Peano's proof consisted in the expression of the given system of differential equations in terms of complexes and in the use of an approximation process in complex form: Peano was the first to write a *system* of n linear differential equations as a *single* vectorial equation

[21] For the context of Peano's work, see [Crowe 1967] and [Borga et al. 1985, pp. 177-182, pp. 189-198].

$\frac{d\overline{x}}{dt} = \alpha\overline{x}$, where $\overline{x} \in \mathbb{R}^n$ and α is the matrix of coefficients by which is represented the linear operator $\alpha \in \mathscr{L}(\mathbb{R}^n)$.

Let us summarize Peano's proof. Let $\overline{a} \in \mathbb{R}^n$ be an arbitrarily chosen constant n-tuple, and let $\overline{a'} = \int_{t_0}^t \alpha\overline{a}\,dt$, $\overline{a''} = \int_{t_0}^t \alpha\overline{a'}\,dt$, $\overline{a'''} = \int_{t_0}^t \alpha\overline{a''}\,dt \dots$ The components of $\overline{a}$, $\overline{a'}$, $\overline{a''} \dots$ are precisely the numbers introduced in the statement of the theorem. The functions (α_{ij}) are continuous and bounded on (p, q); hence $\|\alpha\|_{\mathscr{L}}$ is also continuous and bounded in (p, q). So, if $M = \max \|\alpha\|_{\mathscr{L}}$ on this interval, we have

$$\|\overline{a}'\| < M\|(t - t_0)\| \cdot \|\overline{a}\| \tag{5}$$

$$\|\overline{a''}\| < M^2 \frac{\|(t - t_0)^2\|}{2!} \cdot \|\overline{a}\|, \dots \tag{6}$$

Now the series

$$\|\overline{a}\| + M\|(t - t_0)\| \cdot \|\overline{a}\| + M^2 \frac{\|(t - t_0)^2\|}{2!} \cdot \|\overline{a}\|, \dots \tag{7}$$

is uniformly convergent throughout (p, q), and its sum is

$$e^{M\|(t-t_0)\|}\|\overline{a}\|. \tag{8}$$

Applying the theorem on term by term differentiation to the series that expresses $\overline{x} = \overline{a} + \overline{a'} + \overline{a''} + \overline{a'''} + \dots$ Peano deduces that $\overline{x}$ is a solution of the given equation and that, for $t = t_0$, $\overline{x} = \overline{a}$. Finally, substituting the values of $\overline{a'}$, $\overline{a''} \dots$ in the development of $\overline{x}$, the series that gives $\overline{x}$ can be expressed by the integral form:

$$\overline{x} = \left(1 + \int_{t_0}^t \alpha\,dt + \int_{t_0}^t \alpha\,dt \int_{t_0}^t \alpha\,dt + \int_{t_0}^t \alpha\,dt \int_{t_0}^t \alpha\,dt \int_{t_0}^t \alpha\,dt + \dots\right)\overline{a}. \tag{9}$$

According to Gantmacher it was Peano who first gave this representation of the resolvent matrix [Gantmacher 1959c, p. 127].

In the French translation of his article, Peano introduces the notation $E\left(\begin{smallmatrix} t \\ t_0 \end{smallmatrix}\right)$ and obtains the following expression of the solution to the given system:

$$\overline{x}_t = \overline{a} + \int_{t_0}^t \alpha\overline{a}\,dt + \int_{t_0}^t \alpha\,dt \int_{t_0}^t \alpha\overline{a}\,dt + \dots = E\begin{pmatrix} t \\ t_0 \end{pmatrix} \overline{a} \tag{10}$$

where $\overline{a}$ is the constant n-tuple. The symbol $E\binom{t}{t_0}$ represents an operator $E : \mathscr{L}(\mathbb{R}^n) \times \mathbb{R} \times \mathbb{R} \to \mathscr{L}(\mathbb{R}^n)$, where each member is obtained by iterating the integration process which satisfies the following properties[22]:

$$E\begin{pmatrix} t_0 \\ t_1 \end{pmatrix} E\begin{pmatrix} t_1 \\ t_0 \end{pmatrix} = 1 \tag{11}$$

$$E\begin{pmatrix} t_2 \\ t_0 \end{pmatrix} = E\begin{pmatrix} t_2 \\ t_1 \end{pmatrix} E\begin{pmatrix} t_1 \\ t_0 \end{pmatrix} \tag{12}$$

$$\frac{dE\begin{pmatrix} t_1 \\ t_0 \end{pmatrix}}{dt_1} = \alpha E\begin{pmatrix} t_1 \\ t_0 \end{pmatrix} \tag{13}$$

$$-\frac{dE\begin{pmatrix} t_1 \\ t_0 \end{pmatrix}}{dt_0} = E\begin{pmatrix} t_1 \\ t_0 \end{pmatrix} \alpha. \tag{14}$$

At the end of this note Peano considers systems of non-homogeneous linear differential equations and systems of homogeneous and non-homogeneous differential equations with constant coefficients. In particular he remarks that, for $t_0 = 0$, the solution for the last kind of systems is[23]:

$$\overline{x} = \left(1 + \alpha t + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \dots\right) \overline{a} = e^{\alpha t}\overline{a}. \tag{15}$$

Peano's note "*Integrazione per serie...*" was soon reviewed by Hamburger in the *Jahrbuch über die Fortschritte der Mathematik* [Hamburger 1887, pp. 308-309; 1888, p. 329] and was cited by many mathematicians in Italy

[22] Today these relations are usually called 'semi-group properties'.

[23] In short, we can say that Peano finds here, in complete analogy to the one-dimensional case, the proof of the following theorem: let A be a $n \times n$ matrix, then the series $U(t) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = e^{At}$ converges for all $t \in \mathbb{R}$, and $t \mapsto \overline{x}(t) = U(t)\overline{x_0} = e^{At}\overline{x(0)}$ is the unique solution of the system $\frac{d\overline{x}(t)}{dt} = A\overline{x}(t)$ with the initial condition $\overline{x(0)} = \overline{x_0}$.

and elsewhere[24]. At the fourth International Congress of Mathematicians held in Rome in April 1908, E.H. Moore declared:

"Peano in 1888 in vol. 22 of the *Mathematische Annalen* gave an elegant treatment of the corresponding theorem for a system on n equations: $D_t\gamma_{it} = \sum_{j=1}^{j=n} x_{ijt}\rho_{jt}$, $(i = 1, 2, \ldots, n)$, in the n-partite function $(\gamma_1, \ldots, \gamma_n)$ of P, with the initial conditions: $\gamma_{it_0} = a_i$ $(i = 1, 2, \ldots, n)$" [Moore 1909, p. 113].

Nevertheless Peano's work did not enjoy a great readership at the time of its publication, for several external and internal reasons: the use of the Italian language, its publication in the *Proceedings* of the Turin Academy (not widely circulated) and the particular symbolism and mathematical techniques. We agree with D. Tournès who stressed the historical merits of Peano's notes:

"Il s'agit, pour la première fois, d'une démonstration d'existence ne laissant rien à désirer [...]. Ce travail de Peano apparaît ainsi comme l'ultime avatar des idées semées vers 1830 par Liouville et Cauchy" [Tournès 1997, p. 289].

Yet it must also be said that the use of concepts, such as those of transformations and their norm, which were not yet mastered at the time and were expressed in an abstract language, deterred the international community of mathematicians from their use. The French translation of Peano's article in *Mathematische Annalen* seems to have been largely ignored, so much so that for many years the paternity of the method of successive approximations was attributed to É. Picard and E. Lindelöf who used a similar procedure for *general* differential systems from 1890 on. Thus Peano felt the need to reaffirm the priority and the independence of his results[25]. By 1908, Peano's contribution had not yet been adequately recognized, at least in his own view and he added the following remark in the last edition of his *Formulario Mathematico*:

"The method of successive approximations has been used in Astronomy for a long time. Cauchy shows the previous theorem in a particular case (Moigno

[24] Among the mathematicians who mentioned Peano's results of 1887 and 1888 we find [Pringsheim 1898, p. 2, p. 26, p. 48, p. 49, p. 54, p. 57, p. 66, p. 67, p. 72, p. 73, p. 77, p. 83, p. 92], [Fubini 1929-1930, p. 6], [Levi 1932, pp. 258-259], [Boggio 1932-1933, pp. 442-443], [Cassina 1933, pp. 330-331], [Volterra & Hostinsky 1938, p. 224], [MacDuffee 1946, p. 99], [Levi 1955, pp. 16-18], [Gantmacher 1959a, pp. 152-153], [Gantmacher 1959b, pp. 111-112], [Tricomi 1962, p. 29, p. 31]. The historical relevance of Peano's articles is underlined in some historical studies: [Viola 1987, pp. 34-35], [Kennedy 1980, pp. 17-18], [Ghizzetti 1986, pp. 54-57] and [Roero 2004, pp. 123-125].

[25] Cf. [Peano 1892, pp. 79-82] and [Peano 1897, pp. 9-18]. For the results of Picard and Lindelöf, cf. [Tournès 1997, pp. 289-294].

Traité t. 2 p. 702). Caqué JdM. a. 1864 t. 9 p. 185, and Fuchs AdM. a. 1870 t. 4 p. 36 extend it to other cases. I stated it in a general way in: Torino A. a. 1887, MA. a. 1888 t. 32 p. 450, TA. a. 1897. See: Encyclopädie t. 2 p. 199. M. Bôcher, American T. a. 1902 t. 3 p. 196. Picard JdM. t. 6 a. 1890 p. 145 gives new applications of this theorem" [26].

The relevance of Peano's results lies chiefly in his preliminary setting of the theory of linear operators, to which more than half of the note is devoted. Peano, a logician, was very interested in this sort of research and later he developed this theory and its applications, in his long essay *Calcolo geometrico* [Peano 1888a]. His work on systems of differential equations can be seen as early stages in presenting a fundamental part of analysis as a field of application for mathematical symbolism and intrinsic calculus.

Peano continued his study of complexes and transformations after 1888. He extended and presented these topics, as well as the method of successive approximations, by logical ideography in "*Démonstration de l'intégrabilité des équations differentielles ordinaires*" and in "*Generalità sulle equazioni differenziali ordinarie*"; these topics appeared in all five editions of the *Formulario*, together with bibliographical and historical details, and they were described in part in Peano's lectures at the Turin Military Academy[27].

We can date as of 1887 the idiosyncratic turn towards the application of symbols and methods of mathematical logic to the study of analytic problems. The method of successive approximations – as Peano himself remarked – was well-known at the end of the nineteenth century, but the formalization given by him revealed both novelty and originality. Its

[26] [Peano 1908, pp. 432-433]: "*Methodo de approximationes successivo occurre in Astronomia, ab longo tempore. Cauchy expone theorema praecedente in casu particulare* (Moigno Traité t. 2 p. 702). Caqué JdM. a. 1864 t. 9 p. 185, et Fuchs AdM. a. 1870 t. 4 p. 36 *extende ad alios casu. Me enuntia illo in generale in*: Torino A. a. 1887, MA. a. 1888 t. 32 p. 450, TA. a. 1897. *Vide*: Encyclopädie t. 2 p. 199. M. Bôcher, American T. a. 1902 t. 3 p. 196. Picard JdM. t. 6 a. 1890 p. 145 *da novo applicationes de theorema*".
The works cited here by Peano are the following: [Moigno 1844, p. 702], [Caqué 1864, pp. 185-222], [Fuchs 1870, pp. 36-49], [Peano 1887, pp. 437-446], [Peano 1888b, pp. 450-456], [Peano 1897, pp. 9-18], [Painlevé 1900, p. 199], [Bôcher 1902, pp. 196-215] and [Picard 1890, pp. 145-210].

[27] Cf. [Peano 1890, pp. 186-187], [Peano 1888a, pp. 141-170], [Peano 1893, vol. 2, pp. 1-5, p. 40 ssg.], [Peano 1897, pp. 9-18], [Peano 1895, pp. 60-61], [Peano 1899, pp. 137-139], [Peano 1901, pp. 160-174], [Peano 1903, pp. 213-215, pp. 330-331] and [Peano 1908, pp. 149-151, pp. 432-433].

rigour, elegance and simplicity were remarkable. The new abstract procedure, based on a functional approach, was also explicitly recognised by Pincherle when he described Peano's contributions:

> "Il nous reste enfin à citer quelques travaux qui regardent encore le calcul fonctionnel, mais qui s'en occupent à un point de vue nouveau, qui permet de rendre très claires et presque intuitives certaines généralités de ce calcul. [...] Dans cet ordre d'idées, M. Peano a écrit quelques pages très intéressantes, où, d'une façon aussi sobre que claire, il donne les propriétés les plus simples des opérations distributives appliquées à des éléments déterminés par n coordonnées: on peut dire que c'est une esquisse de la théorie des opérations fonctionnelles distributives exécutées sur les fonctions d'un ensemble linéaire n fois infini [...]" [Pincherle 1897, p. 330].

More recently T. Viola emphasized the logical and general approach chosen by Peano as follows:

> "The methodology of such a formal reasoning is a bold forerunner of what the school of Bourbakism and other analysts keen on abstractionism would have achieved many years later, with a much more general language"[28].

It was this atypical program of research in analysis that condemned Peano's works to isolation until 1910[29].

3. GRAMEGNA ON INFINITE SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Nearly twenty years after Peano's seminal work, Gramegna took up the method of successive integrations. In 1910, Peano suggested to her that she try to generalize the previous theorem to systems of infinite linear differential equations and to integro-differential equations. Gramegna treated this extension in the above-mentioned note "*Serie di equazioni differenziali lineari ed equazioni integro-differenziali*", which opens with the words:

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[28] [Viola 1987, p. 35]: "*La metodologia di un simile procedimento formale è un'ardita anticipazione di quanto sarebbe stato fatto molti decenni più tardi, con liguaggio assai più generale, dalla scuola dei Bourbakisti e da altri analisti appassionati di astrattismo*".

[29] Cf. [Monna 1973, p. 133] and [Guerraggio 1987, pp. 188-192]. The logical direction taken by Peano in his analytic studies had already been criticized by F. Brioschi, E. Beltrami, S. Pincherle, A. Tonelli and V. Volterra in their report for the promotion to full Professor of Infinitesimal Calculus: cf. "*Relazione della Commissione incaricata di giudicare sul concorso alla cattedra di professore straordinario di calcolo infinitesimale nella R. Università di Torino*", *Bollettino ufficiale dell'Istruzione*, XVIII, N. 16, 16.4.1891, p. 428.

"My purpose is to extend this theory for $n = \infty$. Depending on the way we pass to infinity, the right side of the given equations becomes infinite series or integrals and we shall have a series of infinite linear differential equations or an integro-differential equation" [30].

The idea of the extension from finite to infinite systems of linear differential equations surely came from Peano who had an interest in this kind of research since 1888, as Pincherle mentioned when referring to his *Calcolo geometrico*:

"L'auteur [Peano] note encore, sans y insister, qu'on pourrait aussi considérer des systèmes linéaires à un nombre infini de dimensions. Le même point de vue se retrouve dans un travail étendu de M. Carvallo, dont la première partie considère les substitutions linéaires comme des opérations (l'auteur dit opérateurs) appliquées aux vecteurs de l'espace ordinaire: la seconde partie, qui traite de calcul différentiel de ces opérations, rentre comme cas particulier, dans l'ordre des considérations de M. Volterra que nous avons rappelées cidessous" [Pincherle 1897, p. 330]; cf. also [Peano 1894, p. 136].

Some mathematicians, Poincaré among them, already dealt successfully with infinite systems, but they considered algebraic infinite systems in particular and some special cases of integro-differential equations in connection with studies of mathematical physics.

Gramegna used instead very general methods and her paper gives an impression of modernity due to her extensive, useful application of the symbolic language. On the other hand, it is indeed the appeal to Peano's ideography that makes it somewhat difficult to understand her work, and so it is not surprising that this article was not widely read at the time. It garnered a weak reaction in Italy and abroad though Gramegna's work was quoted as follows by Peano in the note "*Importanza dei simboli in Matematica*":

"Professor Moore of the University of Chicago applied the symbolism of mathematical logic in his studies of the new problem of integro-differential equations in a communication to the Fourth International Congress of Mathematicians held in Rome in 1908, and then in his book *Introduction to a Form of General Analysis*, 1910. The same method was applied by Doctor Maria Gramegna, a victim of the earthquake in Avezzano in January of this year, in the paper *Serie*

[30] [Gramegna 1910, pp. 469-470]: "*Mi propongo di estender questa teoria ad n infinito. Dipendentemente dal modo di passare all'infinito, i secondi membri delle equazioni date diventano serie infinite oppure integrali, e si avrà o una serie di infinite equazioni differenziali lineari, o un'equazione integro-differenziale*".

di equazioni differenziali lineari, published in 'Atti della R. Acc. delle Scienze di Torino', 13th March, 1910" [31].

In order to extend the theory developed by Peano, Gramegna started by examining the "theory of *substitutions*, or linear functions of infinite variables, or homographies (*omografie*), in an infinite dimensional space" [Gramegna 1910, p. 470].

The sources of Gramegna's research are in the works of her teacher, Peano: the article "*Integrazione per serie delle equazioni differenziali lineari*" [1887; 1888b] and the last edition of the *Formulario mathematico* [1908], but also in then recent research literature, suggested and analysed by Peano in his University lectures of Higher Analysis. Among the papers cited by her we note works on integro-differential equations by Fredholm [1903], Hilbert [1904-1912, reprint in 1912] and Volterra [1910b], as well as the book by Moore [1910]. Gramegna was inspired, as were Volterra, Fredholm and Hilbert, by the analogy with the theory of systems of linear equations in algebra, as well as by the approach based on abstract linear spaces which had been developed in Italy by Peano and Pincherle [Pincherle 1897; 1901]. There are connections between Gramegna's note and works by Fréchet [1906; 1907] and Moore [1910] in general analysis. In this sense her point of view is quite different from that set out by Riesz in his book *Les systèmes d'équations linéaires à une infinité d'inconnues* [Riesz 1913, pp. 78-121, pp. 156-171] and it provides an interesting example that fills the "vacuum in the theory of linear spaces in the years from 1900 (or 1888) to about 1920" [Monna 1973, pp. 133-134]. Gramegna's article is also interesting because there are practically no references to Italian studies in the works of Hilbert, Schmidt, Riesz, Fréchet or Banach, so it seems that the line of research set by Peano and Pincherle was broken off after 1900. Gramegna's paper shows an attempt to link new trends coming from other countries to the intrinsicly Italian approach in functional analysis. This notwithstanding, Gramegna's work had little following and, as Monna surmises:

[31] [Peano 1915, p. 172]: "*Il Prof. Moore dell'Università di Chicago, ha applicato il simbolismo della logica matematica a studiare il nuovo problema delle equazioni integro-differenziali, in una comunicazione nel 4º Congresso Matematico Internazionale di Roma nel 1908, e poi nel libro* Introduction to a Form of General Analysis*, 1910. Lo stesso metodo fu applicato dalla Dottoressa Maria Gramegna, vittima del terremoto di Avezzano nel gennaio di quest'anno, nello scritto* Serie di equazioni differenziali lineari*, in 'Atti della R. Acc. delle Scienze di Torino', 13 marzo 1910*".

"For many years after Peano (1888) and Pincherle ($\pm$1890) defined linear transformations in an intrinsic way as linear (or distributive) operators, the mathematicians continued to introduce them in articles and books as linear substitutions, that is in the form of matrices" [Monna 1973, p. 133].

Let us take a closer look at Gramegna's work. First, Gramegna introduces the notion of *infinite complex*, that is a bounded sequence $a = (a_n)_{n\in\mathbb{N}}$, and denotes the space of such complexes as C_∞. We will use modern notation in order to make her approach more understandable, as her terminology is now obsolete. Whereas Peano's notes of 1887-1888 were presented in a language that is close to the modern one, Gramegna applies very strictly the logical symbolism developed by Peano during the years 1887-1908. Some of the symbols she adopted have changed meaning or are forgotten today. This is the case, for example, for the logical notations ', $|$, $.\supset.$, $\sim$, Cls, ι, as well as for the analytic ones: l', C_∞, $\mathrm{Subst}\,\mathrm{C}_\infty$, sgn, S, etc. We will use $\Longrightarrow$ instead of $.\supset.$, the symbol 'sup' instead of l', the integration symbol $\int$ instead of S, and so on. In particular cases a reconstruction of the meaning of an entire symbolic definition will be necessary. This is the case, for example, for the notation C_∞, which Gramegna introduced :

"We shall especially consider bounded sequences, and we shall call them 'complexes of infinite order' or 'infinite complexes' and we shall denote their space by C_∞, i.e: $x \in \mathrm{C}_\infty .=. x \in \mathrm{qFN}_1 . \mu x \in \mathrm{Q}$. Def." [32].

To make the deductive procedure simpler, we will denote this Banach space by the standard notation

$$(1) \qquad \ell^\infty(\mathbb{R}^n) = \ell^\infty = \Big\{ a = (a_j)_{j\geq 1} : \sup_{j\geq 1} |a_j| < \infty \Big\}.$$

After having introduced infinite complexes, Gramegna defines *equality* for infinite complexes, and the operations of *sum* and *scalar product*, naturally extending the exposition given by Peano. She then recalls the three definitions, introduced respectively by Peano, Jordan and Leibniz, for the *modulus*, the *deviation* or *scarto*, and the *mole* of an n-tuple $\overline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ [Gramegna 1910, pp. 470-472]. Using

[32] [Gramegna 1910, p. 472]:"*Noi considereremo in modo speciale le successioni limitate, e le diremo semplicemente 'complessi d'ordine infinito' o 'complessi infiniti' e li indicheremo con* C_∞*, cioè:* $x \in \mathrm{C}_\infty .=. x \in \mathrm{qFN}_1 . \mu x \in \mathrm{Q}$. *Def.*".

modern notations, these concepts may be defined as[33]:

$$\operatorname{mod}(a) = |a| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}, \tag{2}$$

$$\operatorname{scarto}(a) = \sum_{i=1}^{n} |a_i| \tag{3}$$

and

$$\operatorname{mole}(a) = \max_{1 \leq i \leq n} |a_i| \;^{34}. \tag{4}$$

These correspond to the Euclidean norm, and to the p-norm for $p = 1$ and $p = \infty$ respectively.

She observes that all these norms are topologically equivalent in $\mathbb{R}^n$. In her study, however, she uses only the extension of the concept of *mole* which, currently speaking, corresponds to the notion of the sup-norm

$$\mu a = \|a\|_\infty = \sup_{j \geq 1} |a_j|. \tag{5}$$

This paragraph is particularly interesting because Peano had not introduced different norms in his articles, nor had he considered the problem of their equivalence in finite dimension, so here Gramegna went beyond her mentor's work.

Using the same method as Peano, Gramegna defines the *limit*, the *derivative* and the *integral* for a bounded sequence with respect to a variable t [Peano 1908, pp. 144-145, p. 284], as well as the notion of convergence in the sup-norm. She then considers linear operators. A *substitution*

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33 In order to preserve Gramegna's notations as much as possible, in the following paragraphs I shall denote the modulus of a complex with the symbol $|\cdot|$. I have also corrected evident typographical errors in Gramegna's note.

34 In Peano's language, it is:

$$n \in \mathrm{N}_1 \,.\, x \in \mathrm{C}_n \,.\supset.\, \mathrm{m}x = \sqrt{}\,[\textstyle\sum(x_r^2 | r,\, 1 \cdots n)];$$
$$x \in \mathrm{C}_n \,.\supset.\, \mathrm{Scarto}\, x = \textstyle\sum[\mathrm{m}x_r | r,\, 1 \cdots n];$$
$$\mu x = \max(\mathrm{m}x_r | r\text{`}\, 1 \cdots n).$$

For the introduction of these terms, cf. [Gramegna 1910, p. 470]. In [Peano 1908, p. 94], we read: "mod a, read 'modulo de a' [...] corresponds to the notation mol a 'moles a' in Leibniz; and the notation that we have introduced here is in Argand a. 1814, in Cauchy, ..." ("*mod a, lege 'modulo de a' [...] habe forma mol a, 'moles a' in Leibniz; et forma que nos adopta in Argand a. 1814, in Cauchy,* ..."). About the introduction of the 'écart', see C. Jordan, *Cours d'analyse de l'École polytechnique*, t. I, Paris: Gauthier-Villars, 1893, pp. 18-19.

or a *homography for infinite complexes* is an operator defined on ℓ^∞

$$A : \ell^\infty \longrightarrow \ell^\infty \qquad x \longmapsto Ax \tag{6}$$

satisfying the following conditions:

$$\forall x, y \in \ell^\infty \quad A(x+y) = A(x) + A(y) = Ax + Ay \tag{7}$$

$$\forall x \in \ell^\infty \quad \sup_{\|x\|_\infty \le 1} \|Ax\|_\infty < \infty \quad \text{(this sup is a finite number)} \tag{8}$$ [35].

The space of linear operators defined on ℓ^∞ is denoted Subst C_∞ and corresponds to the set $\mathscr{L}(\ell^\infty)$. Gramegna remarks that, in this space, it is possible to define the usual algebraic operations of *sum*, *product* and *power* with positive integer exponent, thus extending the definitions given by Peano for an operator defined on $\mathbb{R}^n$ [Peano 1908, pp. 149-150].

She then defines the *mole of a substitution*[36] $A \in \mathscr{L}(\ell^\infty)$, which we now call the operator norm on $\mathscr{L}(\ell^\infty)$, in the following way[37]:

$$\|A\|_{\mathscr{L}} = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \tag{9}$$

and she recalls the properties:

$$\forall x \in \ell^\infty \; \forall A \in \mathscr{L}(\ell^\infty) \quad \|Ax\| \le \|A\| \cdot \|x\|, \tag{10}$$

$$\forall A, B \in \mathscr{L}(\ell^\infty) \quad \|A+B\| \le \|A\| + \|B\|, \tag{11}$$

$$\forall A, B \in \mathscr{L}(\ell^\infty) \quad \|AB\| \le \|A\| \cdot \|B\|. \tag{12}$$

With these tools Gramegna introduces the *exponential* of a bounded operator[38] as

$$e^A = 1 + A + \frac{A^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{13}$$

and proves the absolute convergence of the exponential series.

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[35] One can show that the conditions (7) and (8) imply the property of homogeneity: $A(\lambda x) = \lambda A(x)$ for each $\lambda \in \mathbb{R}$ and $x \in \ell^\infty$. Therefore A is a linear operator.

[36] In Peano's symbols it is $\mu a = \mathrm{l}'\left[\frac{\mu a x}{\mu x} \,\middle|\, x ‘ C_\infty \sim \iota 0\right]$.

[37] In the following paragraphs we omit the specification of the norm we are referring to, because the context does not present ambiguity.

[38] Like Peano, Gramegna considers only continuous (bounded) operators.

Finally Gramegna introduces the notion of *kernel of a bounded substitution*: let $A \in \mathscr{L}(\ell^\infty)$, $s \in \mathbb{N}$, and $i_s = (0,0,\dots,0,1,0,\dots)$ in ℓ^∞ a sequence belonging to the canonical basis defined on ℓ^∞ [39]. Then, an r-component of the sequence $Ai_s \in \ell^\infty$ is denoted by $(Ai_s)_r = u_{rs}$. Let us now consider the matrix

$$u = (u_{rs})_{r,s=1,\dots,\infty} = \begin{pmatrix} u_{11} & u_{12} & \dots \\ u_{21} & u_{22} & \dots \\ & \dots & \end{pmatrix}, \tag{14}$$

infinite in two directions, as a real function of two variables such that

$$\begin{aligned} u : \mathbb{N} \times \mathbb{N} &\longrightarrow \mathbb{R} \\ (s,r) &\longmapsto u(s,r) = (Ai_s)_r = u_{rs} \end{aligned} \tag{15}$$

The matrix u is called *kernel*[40], or *characteristic function* or *generating function* and Gramegna denotes by su the linear operator defined on ℓ^∞ and represented by the matrix u. Then sux stands for a system of infinite equalities such that

$$sux = su(x) = ((sux)_1, (sux)_2, \dots), \text{ where} \tag{16}$$

$$\begin{cases} (sux)_1 = u_{11}x_1 + u_{12}x_2 + \dots \\ (sux)_2 = u_{21}x_1 + u_{22}x_2 + \dots \\ \qquad\qquad \dots \\ \qquad\qquad \dots \end{cases}$$

Gramegna shows that a necessary and sufficient condition for having $su \in \mathscr{L}(\ell^\infty)$ is that all the series on the right side be convergent and $\sup_r \sum_{s\in\mathbb{N}} |u_{rs}| < \infty$.

Applying these notations, she proves the following theorem [Gramegna 1910, pp. 479-480] [41]: consider an infinite system of differential equations

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[39] About the introduction of the canonical basis cf. [Gramegna 1910, p. 473].

[40] For the philological meaning of this term, which was used by Hilbert and taken up again by Peano in his University lectures, Gramegna refers to [Peano 1909, pp. 38-39].

[41] "*Allora abbiasi un sistema di infinite equazioni differenziali con infinite incognite*

$$\begin{cases} \frac{dx_1}{dt} = u_{11}x_1 + u_{12}x_2 + \dots + u_{1n}x_n + \dots \\ \frac{dx_2}{dt} = u_{21}x_1 + u_{22}x_2 + \dots + u_{2n}x_n + \dots \\ \dots \end{cases}$$

in an infinite number of unknowns:

$$\begin{cases} \dfrac{dx_1}{dt} = u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n + \ldots \\ \dfrac{dx_2}{dt} = u_{21}x_1 + u_{22}x_2 + \cdots + u_{2n}x_n + \ldots \\ \ldots \end{cases} \tag{17}$$

where every u_{rs} is constant with respect to time. Let A be the substitution represented by the matrix of the u_{rs} and $\|su\| < \infty$, let x be the sequence $(x_1, x_2, \ldots)$ and x_0 its initial value. We may write the given differential equations as the single equation $Dx = Ax$, and the integral is given by $x_t = e^{At}x_0$. In other words, the different values of x correspond to the different values of t, applying to the complex x_0 the exponential substitution e^{At}, which has the following representation:

$$1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots \tag{18}$$

In Peano's symbolic language this theorem is set out in the following way:

$$A \in \mathrm{Subst}\,\mathrm{C}_\infty . \supset\!: \; x \in \mathrm{C}_\infty\,\mathrm{Fq}. \, Dx = Ax \; .=. \; x = (e^{At}x_0 | \, t, q). \tag{19}$$

Gramegna gives a "simple" proof. From $Dx = Ax$ she infers that $Dx_t - Ax_t = 0$, so $e^{-At}(Dx_t - Ax_t) = 0$. Hence $D(e^{-At}x_t) = 0$, $e^{-At}x_t = x_0$ and $x_t = e^{At}x_0$.

There follows a similar treatment for a system of infinite linear differential equations:

$$\begin{cases} \dfrac{dx_1}{dt} = u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n + \ldots \\ \dfrac{dx_2}{dt} = u_{21}x_1 + u_{22}x_2 + \cdots + u_{2n}x_n + \ldots \\ \ldots \end{cases} \tag{20}$$

where u_{rs} are continuous functions of t. It is possible to write the system as $Dx_t = A_t x_t$, where $A_t \in \mathscr{L}(\ell^\infty)$ is the continuous operator associated to

dove le u sono costanti rispetto al tempo. Indichiamo con A la sostituzione rappresentata dalla matrice delle u, supposto che rappresenti una sostituzione, cioè $\mu su \in Q$. Chiamo x il complesso $(x_1, x_2, \ldots)$, e sia x_0 il suo valore iniziale. Le equazioni differenziali date si potranno scrivere: $Dx = Ax$. E l'integrale è $x_t = e^{At}x_0$ ossia i diversi valori di x corrispondenti ai diversi valori di t si hanno applicando al complesso x_0 la sostituzione e^{At} cioè la sostituzione $1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$"

the infinite matrix of the u_{rs} and, applying Peano's method of successive approximations, Gramegna obtains the value of x_t for an arbitrary value of t:

$$x_t = x_0 + \int_0^t A_t x_0\, dt + \int_0^t A_t\, dt \int_0^t A_t x_0\, dt + \dots \tag{21}$$

The proof of this assertion shows her ability to use rigorously the concept of convergence in the norm which was not well known at that time. Gramegna reasons as follows: since the $\sup_t\{\|A_t\|\} = m$ is a finite number, the terms of the series on the right side of (21) are less or equal to the terms of the series $\|x_0\| \sum_{k=0}^{\infty} \frac{(mt)^k}{k!} = \|x_0\| e^{mt}$. So the series $\|x_t\|$ is convergent in the norm and the series of the derivatives is

$$A_t\Big[x_0 + \int_0^t A_t x_0\, dt + \dots\Big] = A_t x_t \tag{22}$$

which is also convergent in the norm. Then, Gramegna concludes, the series of the derivatives is the derivative of the series, so $Dx_t = A_t x_t$. Therefore, x_t satisfies the given equation.

The paragraph on systems of homogeneous and non-homogeneous linear differential equations is one of the most difficult in Gramegna's work. She writes the proofs almost entirely with the sole support of the logical language, which makes single passages rather difficult to understand[42].

[42] Gramegna takes into consideration homogeneous and non-homogeneous differential systems, both with linear and constant coefficients, finding the relevant solutions for them. In modern terminology, we can briefly say that she proves, in this first part of the paper, the following theorem: if $A = (a_{ij}) \in \mathscr{L}(\ell^\infty)$, then the series $U(t) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$ converges,

and the function $t \mapsto (x_k(t))_{k\in\mathbb{N}} = U(t)((x_k)_{k\in\mathbb{N}}) = e^{At}((x_k)_{k\in\mathbb{N}})$ is the unique solution of the infinite system of linear differential equations, with assigned initial conditions:

$$\begin{cases} \frac{dx_1(t)}{dt} = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + \dots, & x_1(0) = x_{01} \\ \frac{dx_2(t)}{dt} = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + \dots, & x_2(0) = x_{02} \\ \dots \\ \frac{dx_n(t)}{dt} = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + \dots, & x_n(0) = x_{0n} \\ \dots \end{cases}.$$

Here Gramegna applies the notation $E(A; t_0, t)$, introduced by Peano, to express the solution of the above system[43]:

$$x_t = x_0 + \int_{t_0}^{t} A_t x_0 \, dt + \int_{t_0}^{t} A_t \, dt \int_{t_0}^{t} A_t x_0 \, dt + \cdots = E(A; t_0, t) x_0. \tag{23}$$

The operator E satisfies the properties that Peano had pointed out in the finite-dimensional case: the relevant relations are again stated by Gramegna, without giving proofs. (Regrettably the manuscript of Gramegna's thesis, where they were probably given in full, is lost.) A reconstruction of these proofs can be based on the relevant paragraphs of the fifth edition of *Formulario Mathematico*. Gramegna implicitly uses the global uniqueness of the integral curves of the given system to prove the properties

$$E(A; t_0, t_1) = E(A; t_1, t_0)^{-1} \tag{24}$$

$$E(A; t_0, t_2) = E(A; t_1, t_2)\, E(A; t_0, t_1) \tag{25}$$

Gramegna could have developed the following reasoning: let us consider the equation $\frac{dx}{dt} = A_t x$ and the operator $E(A; t_0, t_1)$, where $A \in \mathscr{L}(\ell^\infty)$. From (23) it follows that E is an operator such that, if $x_0 \neq 0$ and x_1 are the values of a sequence x, which satisfy the given equation $\frac{dx}{dt} = A_t x$, then $x_1 = E(A; t_0, t_1) x_0$. Since $x_0 = E(A; t_1, t_0) x_1$, then $x_0 = E(A; t_1, t_0)\, E(A; t_0, t_1) x_0$. Therefore $1 = E(A; t_1, t_0)\, E(A; t_0, t_1)$. Now, let us consider $x_2 = E(A; t_0, t_2) x_0$ and let us fix $t = t_1 \in [t_0, t_2]$. We can write

$$x_2 = E(A; t_0, t_2) x_0 = E(A; t_1, t_2) x_1 = E(A; t_1, t_2)\, E(A; t_0, t_1) x_0. \tag{26}$$

So, $E(A; t_0, t_2) x_0 = E(A; t_1, t_2)\, E(A; t_0, t_1) x_0$ and $E(A; t_0, t_2) = E(A; t_1, t_2)\, E(A; t_0, t_1)$. Thus, the second relation is also proved.

Finally, Gramegna hints at infinite determinants and she mentions the then recent research by Poincaré and von Koch to show that such a determinant assumes a value[44].

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[43] Gramegna writes the relations by putting $t_0 = 0$, but we prefer to maintain uniformity with Peano's previous treatment. She also writes the notation $E(A; t_0, t)$ on a single line, while Peano used the symbol $E\binom{t}{t_0}$ without specifying the operator.

[44] Cf. [Poincaré 1886] and [Koch 1891].

Let $u = (u_{rs})_{r,s=1,\ldots,\infty}$. Then the determinant D is defined to be the limit (if it exists) of the $n \times n$ determinants $\{u_{rs} : r, s = 1, 2, \ldots, n\}$ as n goes to infinity, that is

$$Du = \lim_{n \to \infty} D_n u. \tag{27}$$

Gramegna recalls Poincaré's conditions (published in 1886) for a determinant of infinite order to have a value: the limit of such a determinant exists if, for each i, $a_{ii} = 1$ and the double series $\sum_{\substack{r,s=1 \\ r \neq s}}^{\infty} u_{rs}$ is absolutely convergent. She also recalls the extension of the criterion given by von Koch in 1891: the limit of the given determinant exists if the product of the terms lying on the main diagonal is absolutely convergent and the double series of the elements outside the diagonal also converges absolutely.

Then, Gramegna states that the determinant of an operator is the determinant of its matrix. If $u = (u_{rs})_{r,s=1,\ldots,\infty}$ and the series $\sum_{\substack{r \in \mathbb{N} \\ s \in \mathbb{N}}} |u_{rs}|$ is convergent, then $D(1 + su)$ assumes a finite value. She calls the coefficients of the development of $D(1 + lA)$ *invariants of the substitution* A, according to the increasing powers of l. The first invariant I – today called *trace* – is given by

$$Iu = u_{11} + u_{22} + \cdots = \sum_{r \in \mathbb{N}} u_{rr} \tag{28}$$

It is thus possible to prove the following theorem:
If $A \in \mathscr{L}(\ell^{\infty})$ and $\mathrm{tr}(a) = IA = \sum_{r \in \mathbb{N}} u_{rr}$ is convergent, then $D(e^A) = e^{IA}$.

4. INTEGRO-DIFFERENTIAL EQUATIONS

In the last section of her article, Gramegna applies both matrix notation and logical symbolism to solve integro-differential equations, which were previously studied by Fredholm and Volterra.

She starts studying integral operators with special consideration for Abel's equation

$$g(x) = \int_0^1 k(x, y) f(y)\, dy \tag{1}$$

where $k \in \mathscr{C}([0,1]^2)$, $f \in \mathscr{C}([0,1])$ and $x \in [0,1]$.

She calls *continuous complex* a real and continuous function defined on $[0,1] \subseteq \mathbb{R}$ and denotes the set of continuous complexes by C_c, that is, using modern notations the function space $\mathscr{C}([0,1])$. Supposing x variable in the interval $[0,1]$, Gramegna calls *continuous substitution* an operator sk defined as follows:

$$\begin{aligned}(2) \qquad \mathrm{sk} : \mathscr{C}([0,1]) &\longrightarrow \mathscr{C}([0,1]) \\ f &\longmapsto \mathrm{sk}\, f = (\mathrm{sk}) f = \int_0^1 k(\cdot, y) f(y)\, dy\end{aligned}$$

for $k \in \mathscr{C}([0,1]^2)$.

Extending her definition of a *mole* and recalling that $\|f\| = \sup_{x\in[0,1]} |f(x)|$, she states that the *mole of a continuous substitution* is given by

$$\|\,\mathrm{sk}\,\| = \sup_{\substack{f\in\mathscr{C}[0,1] \\ f\neq 0}} \frac{\mathrm{sk}\, f}{\|f\|}, \tag{3}$$

and

$$\|\,\mathrm{sk}\,\| = \sup_{x\in[0,1]} \left\{ \int_0^1 |k(x,y)|\, dy \right\}. \tag{4}$$

Further, Gramegna remarks that it is possible to define the *functional product* of two continuous substitutions with kernels $k, h \in \mathscr{C}([0,1]^2)$ and the *exponential* of a substitution as

$$e^{\mathrm{sk}} = 1 + \mathrm{sk} + \frac{(\mathrm{sk})^2}{2!} + \frac{(\mathrm{sk})^3}{3!} + \dots, \tag{5}$$

proving that $\|e^{\mathrm{sk}}\| \le e^{\|\,\mathrm{sk}\,\|}$.

She is now able to study integro-differential equations of the type $\frac{\partial f(t,x)}{\partial t} = \int_0^1 k(x,y) f(t,y)\, dy$, where $k \in \mathscr{C}([0,1]^2)$, $f = f(t,x)$ has to be determined, $t \in \mathbb{R}$, and $x \in [0,1]$. The equation can be written as $Df = (\mathrm{sk}) f$, and its integral is given by[45]

$$\begin{aligned}(6) \quad f(t,x) = e^{t\,\mathrm{sk}} f(0,x) = f(0,x) &+ t \int_0^1 k(x,y) f(0,y)\, dy \\ &+ \frac{t^2}{2!} \int_0^1 k(x,y)\, dy \int_0^1 k(y,z) f(0,z)\, dz + \dots\end{aligned}$$

[45] In Peano's symbolism it is: $k \in \mathrm{qF}(\theta,\theta)\,\mathrm{contFq}\,.\supset\colon f \in \mathrm{C}_c\mathrm{Fq}.\,\mathrm{D}f = \mathrm{sk}\, f\,.=.\, f = [e^{t\,\mathrm{sk}} f0 \,|\, t, q]$.

Finally, she studies the integro-differential equation[46]

$$\frac{\partial f(t,x)}{\partial t} = \int_0^1 k(t,x,y) f(t,y)\,dy + h(t,x), \tag{7}$$

where $k \in \mathscr{C}([0,1]^2)$, $h \in \mathscr{C}([0,1])$ and $f \in \mathscr{C}([0,1])$. The equation can be written in the abstract form $Df = \mathrm{sk}\, f + h$ and, using the notation she introduced before, the integral is given by[47]

$$f = E(\mathrm{sk}; 0, t) f_0 + \int_0^t E(\mathrm{sk}; t, 0) h_t\,dt. \tag{8}$$

Gramegna does not give proofs of her statements and merely remarks that, in all these cases, it is sufficient to repeat the same reasoning as in the first part of the article, in order to study infinite systems of linear differential equations[48]. She does not venture further into the matter, and claims that the topic has been previously treated by Volterra in February 1910[49]. In a footnote she adds the following remark:

> In the note by V. Volterra, "*Questioni generali sulle equazioni integrali ed integro-differenziali*", R. Accademia Lincei, 20th February, 1910, the author shows the treatment by which it is possible to transfer the previous considerations to the case of variable limits and to the more general case of constant limits. Very recently, Moore of the University of Chicago, in his *Introduction to a Form of General Analysis*, 1910, studies integro-differential equations by using the methods and the symbols of mathematical logic (p. 11). Nevertheless I think that all the results presented here are new, as well as the methods used to find them, namely the exponential of a substitution and its mole, which make it possible to prove the convergence of this series, in the usual way" [50].

[46] In this case, the integral operator is called *continuous variable substitution* by Gramegna, who underlines the dependance on t in the kernel k.

[47] The final part of Gramegna's note can be summarized by saying that she proves the theorem: if $Af = \int_0^1 k(\cdot,y) f(y)\,dy$ for $k \in \mathscr{C}([0,1]^2)$ and $f \in \mathscr{C}([0,1)]$, then $U(t) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$ converges, and the function $t \mapsto u(t) = U(t)f$ is the unique solution of the integro-differential equation $\frac{\partial u(t,x)}{\partial t} = \int_0^1 k(x,y) u(t,y)\,dy$ with the initial condition $u(0) = f$.

[48] The article concludes with a paragraph containing some critical remarks on continuous and discontinuous substitutions. This final part, however, is very short and the remarks seem to be made only in passing.

[49] Cf. [Gramegna 1910, p. 490]: "*Un'equazione simile [...] è trattata da V. Volterra nei 'Rendiconti della R. Accademia dei Lincei', 6 Febbr. 1910*" ("A similar equation [...] is treated by V. Volterra in 'Rendiconti della R. Accademia dei Lincei', 6 Febr. 1910").

[50] [Gramegna 1910, pp. 490-491]: "*Nella nota V. Volterra,* Questioni generali sulle equazioni integrali ed integro-differenziali*, 'R. Accademia Lincei', 20 febbraio 1910, l'autore indica il procedimento col quale possono trasportarsi le considerazioni svolte nel caso dei limiti variabili e in quello più*

In order to understand this reference to Moore's *Introduction to a Form of General Analysis*, we may recall what M. Bernkopf writes about this work:

> "Moore, struck by similarities between Hilbert's work on integral equations [...] and the theory behind the solution of (finite or infinite) systems of linear equations, was led to attempt a generalization which would include all these theories. He looked at families of real valued functions, defined a generalized form of convergence for sequences of these functions, and then considered functionals whose domains were those sets of functions. It is evident from the form of Moore's work that he was consciously working in the realm of operator theory" [Bernkopf 1968, p. 311].

Bernkopf also notes:

> "[...] his work is somewhat isolated. [...] It was Moore's aim [...] to establish a broad theory, called by him *General Analysis*, which would include all of the above-mentioned theories as special cases. It cannot be said that Moore had a great deal of success in influencing his colleagues with his plan, for even some fifteen years after Moore's first attempt, Hellinger and Toeplitz could say [...] 'solution theory is not accomplished through such axiomatic formulation..." [Bernkopf 1966-67, p. 45].

Although the influence of Moore's essays on Gramegna is not clear, we can draw a comparison between Gramegna's and Moore's approach[51]. Both were convinced that there should exist an abstract theory that would include the theory of linear (differential) equations in finite dimensional spaces, the theory of infinitely many linear (differential) equations in infinitely many unknowns, and the theory of integral equations. Gramegna focused on problems about infinite systems of linear differential equations and considered only some special cases of function spaces, introducing the appropriate notion of convergence. Moore's approach was more abstract; the American mathematician developed a theory whose elements are members of a set M of real-valued functions $x(s)$, all defined for s belonging to some abstract set S. Then he introduced the notion of *relative convergence*: the sequence of functions $\{x_p(s)\}$ converges to $x(s)$ relative to a function $\rho(s)$ if, for $\varepsilon > 0$, there is an n,

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*generale dei limiti costanti. Recentissimamente il Moore dell'Università di Chicago nell'*Introduction to a Form of General Analysis *1910, tratta coi metodi e coi simboli della logica matematica (pag. 11) le equazioni integro-differenziali. Però mi pare che tutti i risultati contenuti nel presente mio scritto siano nuovi, come pure i metodi per trovarli, cioè l'esponenziale d'una sostituzione e la sua mole, che permette di riconoscere la convergenza assoluta di questa serie, come per le serie comuni*".

[51] For a detailed analysis of Moore's work, cf. [Siegmund-Schultze 1998].

independent of s, such that $p > n$ implies

$$|x_p(s) - x(s)| \leq \varepsilon\rho(s). \tag{9}$$

This notion includes various kinds of convergence in one definition and, for example, if $\rho(s) = 1$ the convergence is the uniform one, as used by Gramegna.

The logical language[52] in Moore's and Gramegna's papers is also quite different, though the American mathematician explicitly referred to the symbols of Peano's *Formulaire de mathématiques* [Moore 1910, p. 150]. The reception of Moore's article was problematic, as was that of Gramegna's work, and Bernkopf appropriately stresses:

> "Also, his works are difficult to read as they use his own symbolic language which was not generally known to others. In short, it was felt that Moore's General Analysis was cumbersome, and that it added no news insights. It solved no new problems and, in general, seemed to be a dead end. Perhaps, too, it was conceptually a little ahead of its time" [Bernkopf 1966-67, p. 46].

5. THE RECEPTION OF GRAMEGNA'S NOTE

Gramegna's work was praised by Peano and by other members of his school. In April 1910, Peano's assistant and collaborator, Giovanni Vacca expressed true appreciation of that article:

> "I was really satisfied to read the work by Miss Maria Gramegna on differential equations and integro-differential equations. It is really important. Prof. Volterra, in Rome, showed me a manuscript of his *Questioni generali* etc. (20 Febr. 1910), and immediately I was surprised by the deep analogy between his procedure and the method of successive integrations. Miss Gramegna's note clarifies the nature of that analogy and the proofs are so easy and the notation is so appropriate, that I think they couldn't be better" [53].

Vincenzo Mago, a fellow student of Gramegna, wrote with admiration in her obituary:

[52] Cf. [Cajori 1929, pp. 303-305] on Moore's logical language.

[53] G. Vacca to G. Peano, [Rome, April 1910] *in* [Osimo 1992, XIII bis]: "*Ho avuto intanto la grande soddisfazione di leggere la nota della Signorina Maria Gramegna sulle equazioni differenziali ed integro-differenziali. È veramente importante. Il Prof. Volterra a Roma, mi aveva fatto leggere il manoscritto delle sue* Questioni generali *etc. (20 Febbr. 1910), e subito io ero rimasto colpito dalla profonda analogia che il suo procedimento aveva con quello delle integrazioni successive. La nota della Signorina Gramegna mette ben in rilievo di qual natura sia questa connessione. E le dimostrazioni rese tanto semplici dalle notazioni veramente felici, non potrebbero essere più belle*".

> "Systems of infinite algebraic equations had already been studied when Maria Gramegna, using a suggestion from Professor Peano, whose dedicated student she had been for three years, decided to establish the solutions for a system of infinite differential equations. In this newly proposed problem, everything had to be clarified: she introduced the complexes of infinite real numbers, the limits, the series, the substitutions among these objects, she studied infinite determinants and then set up a theory in which the given systems had a meaning; thus she was able to work with these systems and, using the same method expounded by Prof. Peano in 1887 for a finite system of equations, she obtained analogous solutions. She linked that theory to the theory of integro-differential equations, developed by Fredholm (and frequently used in recent works regarding Mathematical Physics) with a short account of more complicated systems [...]. Analysis can not ignore the methods she created" [54].

Gramegna's article is an example of the modern application of the operator approach and the use of logical symbolism, as F. Tricomi emphasized:

> "Author of an essay (published in the volume 45, 1909-10 of the 'Proceedings of Turin Academy') where, using Peano's symbols, she anticipates the modern application of the matrix theory to study systems of differential equations. The idea, however, came very probably from Peano" [55].

[54] [Mago 1915, p. 304]: "*Già s'erano considerati sistemi di infinite equazioni algebriche, quando Maria Gramegna, seguendo un consiglio datole dal Prof. Peano, di cui per tre anni era stata assidua alunna, si propose di stabilire le risolventi d'un sistema d'infinite equazioni differenziali. Nel nuovo problema da lei propostosi tutto doveva ancora acquistare un senso: essa introdusse i complessi d'infiniti reali, i limiti, le serie, le sostituzioni fra questi enti, studiò i determinanti infiniti e così andò costruendo una teoria nella quale venivano ad avere significato i sistemi da lei considerati: poté quindi operare con essi; e, seguendo una via analoga a quella che aveva indicato nel 1887 il professor Peano pel caso d'un numero finito d'equazioni, giunse ad analoghe risolventi. Collegò la sua teoria a quella delle equazioni integro-differenziali di Fredholm (di uso frequente in recenti lavori di Fisica-Matematica), accennando poi a sistemi più complicati. [...] L'Analisi non potrà mai trascurare l'argomento da Lei svolto*".

[55] [Tricomi 1962, p. 61]: "*Autrice di una memoria (pubblicata nel t. 45, 1909-10 degli 'Atti dell'Acc.' di Torino) in cui, facendo uso dei simboli del Peano, precorre la moderna applicazione della teoria delle matrici allo studio dei sistemi di equazioni differenziali. L'idea però era, molto probabilmente, del Peano*".

Tricomi's last assertion is rather curious, because it was evident that Peano, as supervisor of her degree thesis, suggested to her the topic and the tools to prove the theorem. Tricomi's appreciation of Gramegna's note is repeated by Kennedy: "Many of those who wrote theses under Peano's direction were women and, according to Terracini, they were not always well prepared, but the 1910 graduate, Maria Gramegna, was one of the most promising. In fact, Peano had presented a long article by her on differential and integral equations to the Academy of Sciences on 13 March, 1910. In it she anticipated the modern application of matrix theory to the study of systems of differential equations; the idea for this probably came from Peano. Her abilities were not to be realized, however, for she went to teach at the Normal School in Avezzano (L'Aquila) and died on 13 January, Gramegna's work did not enjoy a large readership[56] and is not frequently cited in functional analysis during the first half of the 20th century. This is partly due to the difficulties in understanding the introduction of Peano's logical language into advanced analysis. Nevertheless, Gramegna's paper is mentioned by Hellinger and Toeplitz in the prestigious *Encyklopädie der Mathematischen Wissenschaften* [Hellinger, Toeplitz 1927, p. 1478]. G. Vivanti and Volterra refer to it in a general bibliography on integro-differential equations [Vivanti 1916, p. 378]; [Volterra 1930, p. 168]. In addition, a short and precise summary of the note is published by Toeplitz in the *Jahrbuch über die Fortschritte der Mathematik*:

> "The given system of differential equations $\frac{dx_p}{dt} = \sum_{q=1}^{n} U_{p,q}x_q$ $(p = 1, \ldots, n)$ can be solved in a very elegant way using the calculus of substitutions (matrices), and we can form $e^{Ut} = 1 + Ut + \frac{1}{2!}U^2t^2 + \ldots$, where U is the matrix $(u_{p,q})$. The single columns of the matrix thus defined are a complete system of solutions. The author applies this remark to the case of the aforementioned system of differential equations, which is composed of infinite ordinary differential equations, with an infinite number of unknown functions of t $(n = \infty)$. Hence she assumes that the series $\sum_{q=1}^{\infty} |u_{p,q}|$ is convergent for each p and that the limit of the value, independently from p, is fixed. Then, for matrices of this kind it is possible to establish the calculus. These substitutions belong to the same space of infinite variables $x_1, x_2, \ldots$ that is characterized by the fact that $\limsup |x_p|$ is bounded for $p = \infty$" [57].

Gramegna's work is also connected to a vicissitude that Peano experienced in his later academic life, as it is the last research in Higher Analysis

1915, a victim of the earthquake that destroyed that town and killed 96% of its inhabitants" [Kennedy 1980, p. 132].

[56] Recently, C. Perazzoli has confirmed this, writing: "When I was looking for the volume of the 'Atti della Reale Accademia delle Scienze di Torino' containing Gramegna's article in the library of the Department of Mathematics at the University of Rome 'La Sapienza' in 1994, I was utterly surprised to find the respective pages of this volume still connected so that I had to cut them open in order to read the text. I must say that I could not refrain from being touched by the fact that this important article had gone unnoticed until the very day I held it in my hands" [Hahn & Perazzoli 2000, p. 505].

[57] "Man kann das System gewöhnlicher Differentialgleichungen $\frac{dx_p}{dt} = \sum_{q=1}^{n} u_{p,q}x_q$ $(p = 1, \ldots, n)$ sehr elegant lösen, indem man sich des Substitutionen-(Matrizen-)Kalküls bedient und $e^{Ut} = 1 + Ut + \frac{1}{2!}U^2t^2 + \ldots$ bildet, wo U die Matrix $(u_{p,q})$ bedeutet; die einzelnen Kolonnen der so gebildeten Matrix sind lauter Lösungssysteme. Die Verf. überträgt diese Bemerkung auf den Fall eines entsprechenden Differentialgleichungssystems von unendlichvielen gewöhnlichen Differentialgleichungen für unendlichviele unbekannte Funktionen von t $(n = \infty)$. Sie setzt dabei voraus, daß $\sum_{q=1}^{\infty} |u_{p,q}|$ für jedes p konvergiert und dem Werte nach unter einer von p unabhängigen festen Grenze gelegen ist. Für Matrizen

done under his supervision. Peano was in fact dismissed from the course of Higher Analysis immediately after his submission of Gramegna's article to the Turin Academy of Sciences. We can briefly trace the salient phases of this event from the minutes of the Turin Faculty of Sciences. Gramegna's note was presented to the Academy at the session of 13 March 1910, in the presence of D'Ovidio (President of the Academy), Segre, Peano, Somigliana, Jadanza, Naccari, Guareschi, Guidi, Mattirolo, Fusari and Parona[58]. Four days later, on 17 March, a Faculty meeting took place in Turin to which Dean Corrado Segre, Secretary Gino Fano and the full Professors D'Ovidio, Peano, Somigliana, Boggio, Jadanza, Naccari, Guareschi, Parona, Spezia and Mattirolo attended[59]. During this meeting, Segre criticized the teaching of Higher Analysis conducted by Peano over the past two academic years. After having remarked that the merits of Peano in the fields of mathematics and logic were well-known, Segre attacked the critical and logical approaches that marked his lectures and the use of the *Formulario* as the main textbook:

> "Prof. Peano is universally praised for the critical acumen with which he treated questions regarding the foundations of Elementary Mathematics and of Infinitesimal Calculus. He is also universally known, outside the mathematical domain, for the symbolic language that he developed extensively, called 'mathematical logic' [...]. In this language, with the help of several disciples, he edited the well-known '*Formulario di Matematica*'. Now, the two courses held by Prof. Peano in these years are lacking – in my opinion – for reasons which are perfectly comprehensible with respect to what I have just said. They have a fragmentary and discontinuous character; i.e., the various lectures (with some insignificant exceptions) treat disconnected and almost arbitrarily chosen subjects, without ever, or hardly ever, delving into those theories currently defined as Higher Analysis. Every topic is expounded only in connection with the Mathematical Logic or the *Formulario*, as far as they were developed. The *Formulario* is the main textbook for the students of this course in Higher Analysis of our Faculty. Now, this does not correspond to what such a course should be, in my opinion. In this way, clever students cannot progress in advanced research in Higher Analysis; with these methods they will learn only the critical approach,

dieser Art kann sie nämlich den Kalkül einrichten; diese Substitutionen gehören zu demjenigen Raum der unendlichvielen Variabeln $x_1, x_2, \ldots$, der durch die Beschränktheit von $\limsup |x_p|$ für $p = \infty$ charakterisiert ist" [Toeplitz 1910, p. 388].

[58] Cf. [AST-27].

[59] Cf. [ASUT-VII-83, *Verbale* n° 267 of 17 March 1910, pp. 123-127].

of which Prof. Peano is a master, but not the constructive one, which is essential to this discipline" [60].

Peano retorted that, in his course, he introduced very recent topics and stimulated his students to conduct original research, some of which which either were already published or were in press. Peano added that he took into special consideration every subject that might have been useful for students who wanted to teach in secondary schools. His defence of the importance of rigour, which he claimed as "the first, unavoidable feature of every mathematical research" [61] is particularly vibrant. It made him confident in his methods and tools, which ensure as much as possible that the mathematics was free from errors. Despite this reply, the Faculty debate continued with criticisms by D'Ovidio,

[60] "*Il Prof. Peano è universalmente apprezzato per l'acume critico con cui ha trattato le questioni relative ai fondamenti delle Matematiche Elementari e del Calcolo Infinitesimale. Egli è pure universalmente conosciuto, anche fuori dal dominio delle matematiche, per il linguaggio simbolico, da lui ampiamente sviluppato, che vien chiamato 'logica matematica' [...]. Con questo linguaggio egli, in unione ai vari suoi discepoli, ha redatto il noto* 'Formulario di Matematica*'. Ora i due corsi di Analisi Superiore svolti dal Prof. Peano in questi anni peccano, secondo il mio modo di vedere, per ragioni che si spiegano perfettamente con ciò che ho premesso. Essi hanno un carattere frammentario, saltuario, svolgono cioè nelle varie lezioni (tranne eccezioni non rilevanti) argomenti staccati, che sembran scelti a caso, senza che mai, o quasi mai, sia approfondita qualcuna di quelle teorie che comunemente si designano col nome di Analisi Superiore. Si tratta invece qui ciascun argomento solo per quel tanto che la Logica Matematica, o il* Formulario*, quali furono svolti fino ad oggi, possono dare. Il* Formulario *è il principale testo per gli studenti di Analisi Superiore della nostra Facoltà. Ora ciò non corrisponde a ciò che, secondo me, deve essere un tale corso. Non così i giovani di valore possono essere indirizzati a fare ricerche elevate nell'Analisi Superiore. Così non impareranno altro, se non l'indirizzo critico in cui il Prof. Peano è maestro; non l'indirizzo costruttivo, che è essenziale in questa materia*" [ASUT-VII-83, *Verbale* n^{o} 267 of 17 March 1910, pp. 123–124]. Segre's criticisms were repeated, many years later, by Terracini [1968, pp. 40-41]: "When I was a student, Peano held not only the course of Calculus for students of the second year, but also the course of Higher Analysis for students of the second two years. Sincerely, after so many years, I could not say what the content of the course of Higher Analysis was when I attended it; indeed, thinking better, I should say that perhaps there was no content; in fact – as I can tell on the basis of my memories, after so many years – I think that the lectures were reduced – as well as those in Calculus – to skimming through the *Formulario*, dwelling, at times, on certain subjects" ("*Quando io ero studente, Peano teneva non soltanto il corso di Calcolo per gli studenti del second'anno, ma anche quello di Analisi Superiore per quelli del secondo biennio. Francamente, a distanza di tanti anni non saprei più dire quale fosse il titolo del corso di Analisi Superiore dell'anno in cui io lo frequentai; e anzi, ripensandoci su, direi che forse un titolo non lo avesse nemmeno; perché – almeno a quanto posso dire in base ai miei ricordi dopo tanti anni – ritengo che le lezioni si riducessero, al pari di quelle di Calcolo, a sfogliare il* Formulario *fermandosi sopra qualche punto*").

[61] [ASUT-VII-83, *Verbale* n^{o} 267, p. 125]: "*Il rigore è primo, imprescindibile attributo di ogni ricerca matematica*".

Somigliana and Fano. D'Ovidio emphasized that one must not confuse the teaching of the *Magistero* School, which trained the students for teaching in secondary schools, with the topics of the courses of Higher Mathematics, which had to present new theories in order to provide students with many suggestions, research directions and tools for doing advanced studies[62]. Moreover, D'Ovidio remarked that, in every mathematical theory, the inventive or constructive phase precedes the critical one, therefore both trends must be considered, and no professor should neglect intuition in favour of rigour. These criticisms were shared by Fano, who had been Peano's student in the Calculus course and a collaborator on the edition of the *Formulario* in 1894, even though he was member of Segre's school[63]. Somigliana's speech was even harsher, expressing doubts about Peano's skill in developing fundamental chapters of Higher Analysis such as the theory of differential equations and that of elliptic functions[64].

Grieved and disappointed, Peano decided to offer instead a free course of Higher Analysis. It never took place. In this course, he would have carried out his critical study, showing the power of the symbolic language he used in research and teaching[65].

During the meeting of 12 April 1910 the Faculty requested of the National Ministry of Education an examination for the chair of Higher

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[62] Cf. [ASUT-VII-83, *Verbale* n^{o} 267, p. 125].

[63] Cf. [ASUT-VII-83, *Verbale* n^{o} 267, pp. 125-126]: "Prof. Fano, reminding that he was, some years ago, a student of Prof. Peano in the course of Infinitesimal Calculus, remembers and recalls always with great satisfaction the teaching which he gave at that time, and in which the critical approach was contained in a more modest and suitable way. Actually, it is true that the teaching, in the course of time, must follow the new trends that are coming at the scientific world, but, with great and deep regret, he has to add that he does not believe that the changes in Peano's teaching respond to a true interpretation of the new approaches and of the recent improvements [...]" ("*Il Prof. Fano, ricordando di essere stato anni addietro allievo del Prof. Peano nel corso di Calcolo Infinitesimale, ha sempre presente e rammenta con grande soddisfazione l'insegnamento ch'egli allora impartiva, e nel quale il lato critico era contenuto in più modeste e giuste proporzioni. È vero, senza dubbio, che l'insegnamento, col passare del tempo, deve adattarsi alle nuove idee che si fanno strada nel campo scientifico, ma, con vivo profondo rammarico, egli deve pur aggiungere di non credere che i cambiamenti verificatisi nell'insegnamento del Prof. Peano rispondano proprio a una giusta interpretazione di nuove idee e dei progressi degli ultimi anni [...]*").

[64] Cf. [ASUT-VII-83, *Verbale* n^{o} 267, p. 125].

[65] Cf. [ASUT-VII-86]. For the content of this course, cf. [Roero 2004, p. 141].

Analysis in Turin[66]. After having dismissed Peano, in the autumn of 1910, the course was entrusted to G. Fubini, who was professor at the Turin Polytechnic[67]. Notwithstanding some objections, in March 1911 the Faculty withdrew its request for the examination[68]. From this date on, the appointment for the course of Higher Analysis was continually reassigned to Fubini until 1938, when he was forced to leave Italy because of racial laws.

In the spring of 1915, Peano reiterated, for the last time, his dissatisfaction with the appointment of Fubini, stressing that his own scientific merits had been denied, and proposed, without success, that the course be entrusted to his disciple T. Boggio[69].

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[66] Cf. [ASUT-VII-83, *Verbale* n° 268 of 12 April 1910, pp. 128-131]. Peano's colleague N. Jadanza proposed to confirm Peano in the appointment of Higher Analysis for one more year, asking him to do an organic course, with the support of textbooks, in addition to the *Formulario di Matematica*. Nevertheless Segre maintained that the majority of Professors did not approve the approach used by Peano and did not believe that he could submit to a change of his style or follow suggestions on his teaching.

[67] Cf. [ASUT-VII-83, *Verbale* n° 273 of 25 October 1910, p. 145] and [ASUT-VII-83, *Verbale* n° 274 of 15 November 1910, pp. 152-154].

[68] Cf. [ASUT-VII-83, Verbale n° 279 of 18 March 1911, pp. 168-170].

[69] Cf. [ASUT-VII-83, *Verbale* of 11 March 1915]: "In almost all Universities the teaching of Higher Analysis is entrusted for appointment to the Professor of Infinitesimal Calculus, who can train his young students in the third or fourth year for studies and research in the fields of Higher Analysis. Therefore, he considers it less suitable for the advancement of such studies that, after having assigned to him this appointment for the two years 1908-09 and 1909-10, they dismissed him from it, so denying his merits and entrusting it to a person [Fubini] who, in the recent past, showed ignorance of his [Peano's] works and who follows a different approach. Therefore he [Fubini] produces a complete separation between the two courses of Infinitesimal Calculus and Higher Analysis. In the last years he [Peano] refrained from taking part in the Faculty's meetings devoted to confirmations or new proposals of appointments. Today, because he had to be present for other reasons, he can not let this suffered injustice pass in silence. He still remarks that there is in the same Faculty a Professor, Prof. Boggio, who has no appointment: he knows his [Peano's] methods very well and, if the appointment of Higher Analysis should be entrusted to him, he could carry on the teaching of Analysis with the same approach" ("*In quasi tutte le Università l'insegnamento dell'Analisi Superiore è affidato per incarico al Prof.re di Calcolo Infinitesimale, al quale è dato così mezzo di guidare egli stesso, nel 2° biennio, i giovani suoi allievi nello studio e nelle ricerche dei campi più elevati dell'Analisi. Trova perciò poco confacente all'interesse degli studi che, dopo aver dato a lui tale incarico nei due anni 1908-09 e 1909-10, glielo sia stato tolto, disconoscendo i suoi meriti, e affidandolo invece a persona che in passato ha dato prova di non essere al corrente dei di lui lavori, che segue altro indirizzo, e dà luogo così a un completo distacco fra i due insegnamenti di Calcolo Infinitesimale e Analisi Superiore. Negli anni successivi egli si è astenuto dall'intervenire alle adunanze di Facoltà destinate alle conferme o nuove proposte di incarichi: oggi, avendo dovuto per altra*

During the same period, Peano published a list of his works [Peano 1916, pp. 1-8], with comments about their importance and influence on subsequent developments, stressing the relevance of his results in analysis. He recalled the citations of his essays on differential equations in the *Encyclopädie der mathematischen Wissenschaften*, and about his *Formulario Mathematico* he even said:

> "It is a treatise of Infinitesimal Calculus, more complete than my previous ones, which includes the preliminary chapters of Arithmetic, Algebra and Geometry" [70].

This was not the first time that Peano and Segre clashed. In 1891 they engaged in a lively polemic on the *Rivista di matematica*, edited by Peano, in which the different trends that characterized the school of analysis and that of algebraic geometry were pitched against each other[71]. Although Peano and Segre supported two opposed tendencies, both necessary in mathematical research, the distance between these approaches could no longer be seen as the only reason for the clash. Although Segre accused Peano of not being capable of leading the students to discover original results in Higher Analysis, Gramegna's work stands as proof to the contrary. One must also stress that, at the time, Segre was conducting research in collaboration with Fubini who awaited for his promotion to a full professorship, as we learn from his letter to

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ragione intervenire, non può lasciar passare sotto silenzio il torto che gli si è fatto. Fa presente ancora che esiste nella Facoltà anche un altro Professore, il Prof. Boggio, il quale non ha alcun incarico: questi conosce i suoi metodi, e, se a lui fosse affidato l'incarico dell'Analisi Superiore, potrebbe continuare l'insegnamento dell'Analisi secondo il medesimo suo indirizzo").

Although pressure was exerted by the "group" led by Segre, the result of the secret vote about this new proposal was surprisingly balanced. [ASUT-VII-83, *Verbale* of 11 March 1915]: "For the appointment of Higher Analysis the result is the following: Fubini, votes 4; Peano 4; Boggio 2; blank voting-slips 3. Because nobody obtained the absolute majority, a second vote followed, with this result: Fubini votes 4; Peano 4; Boggio 3; blank voting-slips 2" ("*Per l'incarico di Analisi Superiore, la votazione dà il risultato seguente: Fubini, voti 4; Peano 4; Boggio 2; Schede bianche 3. Nessuno avendo riportata la maggioranza assoluta dei voti, si procede a una seconda votazione, col risultato che segue: Fubini voti 4; Peano 4; Boggio 3; schede bianche 2*"). Cf. also [ASUT-VII-83, *Verbale* of 25 March 1915].

[70] [Peano 1916, p. 8]: *"È un trattato, più completo dei miei precedenti, di Calcolo Infinitesimale incluse le parti introduttorie, Aritmetica, Algebra e Geometria*".

[71] Cf. [Segre 1891, pp. 42-65], [Peano 1891a, pp. 66-69] and [Peano 1891b, pp. 156-159]. About this clash cf. also [Giacardi 2001, pp. 155-157] and [Roero 2004, pp. 16-18].

M. Pieri[72]. This could have prompted Segre to attack Peano's teaching publically.

Peano was no longer entitled to train students, and this put an end to his school of analysis in Turin. With sorrow, Peano wrote to his disciple Vacca, on 24 April 1910:

> "I leave the higher teaching, against my will and with sorrow. I gave all my lectures, trying to draw students' attention, and they really became enthused. I tried to live harmoniously with my colleagues, on whom I depend. But they want me to abandon the symbolisms and never talk again about the Formulario and so on. I refused all under these conditions. I taught this course for enjoyment, and not for money. So this is the end. It is unlikely that I will edit another volume of the Rivista. I worked enough, and I have the right to rest, all the more so since colleagues believe my theories dangerous. Someone else can defend the Formulario if they want to. Besides, the book is well enough known, and that never dies. Perhaps I will devote these last years to the interlingua [latino sine flexione] or to gardening... I am a member of the philosophical society of Genova; I enrolled with great ideas, but now I have no more will to work" [73].

[72] Cf. G. Fubini to M. Pieri, Turin 26.11.1909 *in* [Arrighi 1997, p. 66]: "You ask me why I am interested in the chairs of Parma. Here is the plain unvarnished truth. I came to Turin (renouncing the appointment of Analysis), with the almost formal promise that this year I would become full Professor, whether because the Polytechnic has (or at least had) unlimited posts, or because some colleagues were retiring. Now the new law raised a lot of bureaucratic difficulties and legal controversies between the National Ministry and the Polytechnic [...]. In this case, if there is no way out, I already said to my colleagues that if I cannot be full Professor, I will leave at the first opportunity, all the more so because, at the University, I can immediately obtain a full professorship for my long service. That is why, if the full professorship escapes me in Turin, I should probably request a transfer or an examination for one chair in Parma of Algebra or Analytic Geometry, or Calculus (the Calculus is vacant too?)" ("*Tu mi chiedi perché mi interesso alle cattedre di Parma. Eccoti la verità nuda e cruda. Sono venuto a Torino (rinunciando all'incarico di Analisi) dietro promessa quasi formale che quest'anno sarei diventato ordinario, sia perché il Politecnico ha (o meglio aveva) ruoli illimitati, sia perché colleghi andavano a riposo. Ora la nuova legge ha sollevato tante difficoltà burocratiche e questioni legali tra Ministero e Politecnico [...]. Se così è, se non si trova via di uscita, ho già dichiarato ai colleghi che, se non posso diventare ordinario, me ne vado alla prima occasione, tanto più che all'Università il ruolo unico e la mia anzianità mi assicurano un pronto ordinariato. Ecco perché, se l'ordinariato a Torino mi sfugge, mi risolverei probabilmente a chiedere un trasferimento o un concorso a una delle cattedre di Parma: Algebra o Geometria Analitica, o Calcolo (il Calcolo è anche lui vacante?)*").

[73] G. Peano to G. Vacca, Turin 24.4.1910 *in* [Osimo 1992, letter 105]: "*Io abbandono l'insegnamento Superiore, contro la mia volontà e con dolore. Ho fatto tutte le mie lezioni, procurando di interessare gli allievi, che si sono effettivamente interessati. Ho procurato di vivere d'accordo coi colleghi, da cui dipendo. Ma questi vogliono che io abbandoni i simboli, che non parli più del Formulario e altro ancora. Rifiutai ogni conferma in tali condizioni. Facevo quel corso per piacere e non per interesse. Così è finita. Difficilmente farò ancora uscire un volume della Rivista. Ho lavorato*

The same dismay with the incomprehension that surrounded him at the University is indirectly expressed by Peano in his correspondence with Russell:

> "Il y a quelque temps que je me suis retiré de la partie fatigante de l'enseignement. En conséquence, j'ai plus de temps pour moi, mais j'ai moins d'occasion de faire des élèves. Mes anciens élèves se sont répandus dans l'Italie" [74].

In conclusion, Gramegna's note is an appropriate example for the significance of the mathematical research conducted in Peano's school after 1900, even if it is still largely unknown and widely misunderstood.

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abbastanza, ed ho diritto di riposare, tanto più che i colleghi ritengono le mie teorie pericolose. La difesa del Formulario la faccia chi vuole. Del resto esso è un libro già abbastanza noto, e non muore più. Può essere che io dedichi questi ultimi anni all'interlingua o al giardinaggio... Io sono socio della società filosofica di Genova; mi sono iscritto con grandi idee, ma non ho più volontà di lavorare".

[74] G. Peano to B. Russell, Turin 20.3.1912 *in* [Kennedy 1975, p. 219].

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